

ON THE RELATION OF ANTICIPATIVE
STRATONOVICH AND SYMETRIC INTEGRALS:
A DECOMPOSITION FORMULA

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1. Introduction

Let w_t , $t \in [0,1]$, be a standard, F_t -adapted Brownian motion. Let x_t be another, not necessarily adapted, stochastic process and assume that, in some sense to be specified later

$$x_t = \sum_{m=1}^{\infty} x^{(m)} s_t^{(m)},$$

where the random variables $x^{(m)}$ are F_1 measurable random variables and $s_t^{(m)}$ are all generalized-Stratonovich integrable (c.f. definition below). In this note, we give conditions under which

$$\int_0^t x_t \circ dw_t = \sum_{m=1}^{\infty} x^{(m)} \int_0^t s_t^{(m)} \circ dw_t,$$

where \circ denotes generalized Stratonovich integration and the equality is a.s. (cf. lemma 4). We use the criteria we derive to provide some new relations between Stratonovich and Ogawa integrals which do not go through an intermediate chaos decomposition as in [1].

The results below are an outgrowth of some extensions of the Ito lemma pointed out in [2], especially lemma (4.2) there.

We end this introduction with several definitions, adapted mainly from [1] and [3]:

Definition 1 ([3]). Let $0 = \tau_1 < \tau_2 < \dots < \tau_n = T$, and let $\Delta_m = \sup |\tau_i - \tau_j|$. Let y_t be a (not necessarily adapted) stochastic process, and let

$$\bar{y}_k \triangleq \frac{1}{\tau_k - \tau_{k-1}} \int_{\tau_{k-1}}^{\tau_k} y_s ds \quad (1.1)$$

If

$$\sum_{k=1}^n y_k (w_{\tau_k} - w_{\tau_{k-1}}) \xrightarrow[\Delta_n \rightarrow 0]{n \rightarrow \infty} Y^s$$

in probability, and the random variable Y^s does not depend on the particular way in which $\Delta_n \rightarrow 0$ or on a particular choice of the mesh (τ_1, \dots, τ_n) , we call Y^s the "generalized Stratonovich integral" of y_t , and we denote it by $Y^s = \int_0^1 y_t \circ dw_t$. In the sequel, we omit the word "generalized", and all Stratonovich integrals in this paper will be generalized integrals.

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Remark: In [1], the author uses a somewhat different definition of \bar{o} , where y_k in (1.1) is replaced by

$$\bar{y}_k = \frac{y_{\tau_k} - y_{\tau_{k-1}}}{2} ;$$

Those definitions are not equivalent. For remarks concerning this point, cf. the end of section 3.

Definition 2 ([1]). Let y_t be as above, and let $\phi_m(t)$ be a complete orthonormal base in $[0,1]$. Assume that

$$\sum_{m=1}^k \left(\int_0^1 y_s \phi_m(s) ds \right) \int_0^1 \phi_m(s) dw_s \xrightarrow[k \rightarrow \infty]{L^2(\Omega)} Y^0$$

and moreover, assume that Y^0 does not depend on the particular choice of the family $\phi_m(t)$. Then Y^0 is defined to be the Ogawa (or symmetric) integral of y_t , and we denote it by

$$Y^0 = \int_0^1 y_s * dw_s .$$

2. An Approximation Lemma

In this section, we prove our main approximation lemma, namely:

Let $x_t, S_t^{(m)}, m = 1, 2, \dots$ be F_1 measurable stochastic processes, and let $x^{(m)}, m = 1, 2, \dots$ be F_1 measurable random variable.

Assume:

$$S_t^{(m)} \text{ is Stratonovich integrable, } I^{(m)} \triangleq \int_0^1 S_t^{(m)} \circ dw_t \quad (A1)$$

$$\text{Let } \tilde{x}_t^{(M)} \triangleq \sum_{m=1}^M x^{(m)} S_t^{(m)}, \text{ then } (\exists \epsilon > 0) \lim_{M \rightarrow \infty} E \left(\int_0^1 |x_t - \tilde{x}_t^{(M)}| dt \right)^\epsilon = 0 \quad (A2)$$

$$\text{Let } \bar{J}_{(M)}^{(L)} \triangleq \sum_{m=M+1}^L x^{(m)} I^{(m)}, \text{ then } \lim_{M \rightarrow \infty} \lim_{L \rightarrow \infty} \bar{J}_{(M)}^{(L)} = 0 \text{ in prob} \quad (A3)$$

Let $0 = \tau_1 < \tau_2 < \tau_3 < \dots < \tau_N = 1$, and let $\Delta_N = \sup_{i,j} |\tau_i - \tau_j|$.

$$\text{Define } I_N^{(m)} = \sum_{k=1}^N \bar{s}_k^{(m)} (w_{\tau_k} - w_{\tau_{k-1}}),$$

where $\bar{s}_k^{(m)}$ is defined as in (1.1), i.e. $I_N^{(m)}$ is the approximation to $I^{(m)}$ using the N -mesh $\tau_1 \dots \tau_N$.

Let $\tilde{J}_{N(M)}^{(L)} \triangleq \sum_{m=M+1}^L x^{(m)} \tilde{I}_N^{(m)}$. Then

$$(\forall L < \infty) \quad \tilde{J}_{N(0)}^{(L)} \xrightarrow[N \rightarrow \infty]{\text{prob}} \tilde{J}_{(0)}^{(L)} \quad \text{for any N-mesh } \tau_1 \dots \tau_N \text{ when } \Delta_N \rightarrow 0. \quad (\text{A4})$$

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \lim_{L \rightarrow \infty} J_{N(M)}^L = 0, \text{ in probability} \quad (\text{A5})$$

The following easily proved lemma is the basis for our later conclusions:

Lemma 1: Assume (A1) -(A5). Then x_t is Stratonovich integrable, and

$$\int_0^1 x_t \circ dw_t = \lim_{L \rightarrow \infty} \tilde{J}_{(0)}^{(L)} \quad (2.1)$$

Proof. By (A3), $\tilde{J}_{(0)}^{(L)}$ are a Cauchy sequence w.r.t. convergence in probability and therefore they converge in probability to a limit and the R.H.S. of (2.1) is well defined as this limit is probability.

By definition (1), we therefore have to show that

$$\lim_{L \rightarrow \infty} \text{prob} \tilde{J}_{(0)}^{(L)} = \lim_{N \rightarrow \infty} \text{prob} \sum_{k=1}^N \Delta w_k \bar{x}_k \quad (2.2)$$

where $\Delta w_k \triangleq w_{\tau_k} - w_{\tau_{k-1}}$.

We will show that:

$$\tilde{J}_{N(0)}^{(L)} \xrightarrow[N \rightarrow \infty]{\text{prob}} \tilde{J}_{(0)}^{(L)} \quad (2.3a)$$

$$\lim_{L \rightarrow \infty} \lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} \left[\sum_{k=1}^N \Delta w_k \tilde{x}_k^{(M)} - \tilde{J}_{N(0)}^{(L)} \right] = 0 \text{ in prob} \quad (2.3b)$$

and also that

$$\lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} \sum_{k=1}^N \Delta w_k (\bar{x}_k - \tilde{x}_k^{(M)}) = 0 \text{ in prob} \quad (2.3c)$$

Clearly, (2.3) \Rightarrow (2.2). Note that (2.3a) is exactly (A4). To see (2.3b), note that

$$\sum_{k=1}^N \Delta w_k \tilde{x}_k^{(M)} - \tilde{J}_{N(0)}^{(L)} = \sum_{k=1}^N \sum_{k'=1}^M \Delta w_k [(x^{(k')})_{s_k}^{(k')}] - \sum_{k'=1}^L x^{(k')} \left[\sum_{k=1}^N \Delta w_k \bar{s}_k^{(k')} \right]$$

$$= \sum_{k=1}^N \sum_{k'=M+1}^L \Delta w_k X^{(k')-(k')}_{s_k} = \bar{J}_{N(M)}^{(L)}$$

and (2.3b) follows from (A5).

Finally, to see (2.3c), we will show that

$$\exists q > 0 \quad \text{s.t.} \quad \lim_{M \rightarrow \infty} E \left| \sum_{k=1}^N \Delta w_k (\bar{x}_k - \bar{x}_k^{(M)}) \right|^q = 0 \quad (2.4)$$

from which (2.3c), and therefore (2.2), will follow.

Let $z_N \triangleq \max_{k=1, \dots, N} \left| \frac{w_{\tau_k} - w_{\tau_{k-1}}}{\tau_k - \tau_{k-1}} \right|$; then, for $q = \varepsilon/q'$, ε as in (A2),

$$\begin{aligned} E \left| \sum_{k=1}^N \Delta w_k (\bar{x}_k - \bar{x}_k^{(M)}) \right|^q &\leq E(z_N \sum_{k=1}^N \int_{\tau_{k-1}}^{\tau_k} |x_s - \bar{x}_s^{(M)}| ds)^q \\ &\leq E(z_N^{\frac{\varepsilon}{q'-1}})^{\frac{\varepsilon}{q'-1}} E \left(\sum_{k=1}^N \int_{\tau_{k-1}}^{\tau_k} |x_s - \bar{x}_s^{(M)}| ds \right)^{\varepsilon} = E(z_N^{\frac{\varepsilon}{q'-1}})^{\frac{\varepsilon}{q'-1}} E \left(\int_0^1 |x_s - \bar{x}_s^{(M)}| ds \right)^{\varepsilon} \end{aligned} \quad (2.5)$$

For fixed N , the first term in the R.H.S. of (2.5) is finite, being the moment of the maximum (over a finite number) of finite variance zero mean normal variables, which concludes the proof of the lemma.

We specialize the results of lemma 1 to two important particular cases. The first allows one to obtain a sort of "Taylor expansion", similar to the one in lemma (4.2) of [2]. The second will allow us to make connections with the Ogawa integral.

Lemma 2

Let $s_t^{(m)}$ be F_t -adapted continuous semi-martingales, and let their Doob-Meyer decomposition be:

$$s_t^{(m)} = s_0^{(m)} + A_t^{(m)} + M_t^{(m)}$$

where $M_t^{(m)}$ is a continuous martingale and $A_t^{(m)}$ is a continuous bounded variation process.

Let $a_m \triangleq E^{1/2} \|A^{(m)}\|^2$, where $\| \cdot \|$ denotes the total variation norm.

$$b_m \triangleq \sup E^{1/4} \left[\frac{(M_t^{(m)} - M_s^{(m)})^4}{(t-s)^2} \right]$$

$$c_m \triangleq E^{1/2} \int_0^1 (s_t^{(m)})^2 dt$$

and assume that:

$$\sum_{m=1}^{\infty} b_m^2 < \infty, \sum_{m=1}^{\infty} c_m^2 < \infty, \sum_{m=1}^{\infty} a_m^2 < \infty \quad (H1)$$

$$\sum_{m=1}^{\infty} E((x^{(m)})^2) < \infty \quad (H2)$$

$$\sum_{m=1}^{\infty} \left| \sum_{k=1}^N (\tau_k - \tau_{k-1}) E((s_{\tau_{k-1}}^{(m)})^2) - \int_0^1 E(s_t^{(m)})^2 dt \right| < c(N), c(N) \xrightarrow{N \rightarrow \infty} 0 \quad (H3)$$

Then \tilde{x}_t^M converges in $L^1(\Omega \times [0,1])$ (and hence, also in probability), and (A1)-(A5) hold.

Proof. Note that (A1) is trivial here, and that

$$E \int_0^1 |\tilde{x}_t^L - \tilde{x}_t^M| dt \leq \int_0^1 \sum_{m=M+1}^L E |x^{(m)} s_t^{(m)}| dt \leq \sum_{m=M+1}^L (x^{(m)})^2 + \sum_{m=M+1}^L c_m^2$$

which, by (H1), (H2) converges to zero as $L, M \rightarrow \infty$; hence, x_t^M is a Cauchy sequence in $L_1(\Omega \times [0,1])$ and (A2) follows.

Concerning (A5), we write $\tilde{J}_{N(M)}^L$ as:

$$\begin{aligned} \tilde{J}_{N(M)}^{(L)} &= \sum_{m=M+1}^L x^{(m)} \left[\left(\sum_{k=1}^N s_{\tau_{k-1}}^{(m)} (w_{\tau_k} - w_{\tau_{k-1}}) + \sum_{k=1}^N (\bar{A}_k^{(m)} - A_{\tau_{k-1}}^{(m)}) (w_{\tau_k} - w_{\tau_{k-1}}) \right) \right. \\ &\quad \left. + \sum_{k=1}^N (\bar{M}_k^{(m)} - M_{\tau_{k-1}}^{(m)}) (w_{\tau_k} - w_{\tau_{k-1}}) \right] \\ &\triangleq \tilde{J}_{N(M),1}^{(L)} + \tilde{J}_{N(M),2}^{(L)} + \tilde{J}_{N(M),3}^{(L)} \end{aligned} \quad (2.6)$$

To show (A.5), it will be enough to show that for $j = 1, 3$,

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \lim_{L \rightarrow \infty} E |\tilde{J}_{N(M),j}^{(L)}| = 0 \quad (2.7)$$

whereas $\tilde{J}_{N(M),2}^{(L)}$ converges to zero in probability when the limits are taken in the correct order. Note

that

$$E \left[\sum_{k=1}^N s_{\tau_{k-1}}^{(m)} (w_{\tau_k} - w_{\tau_{k-1}}) \right]^2 = \sum_{k=1}^N (\tau_k - \tau_{k-1}) E(s_{\tau_{k-1}}^2)$$

Therefore,

$$\begin{aligned} E |\tilde{J}_{N(M),1}^{(L)}| &\leq \sum_{m=M+1}^L E^{1/2}(x^{(m)})^2 \left(\int_0^1 E((s_t^{(m)})^2) dt \right)^{1/2} \\ &\quad + \sum_{m=M+1}^L E^{1/2}(x^{(m)})^2 \left[\sum_{k=1}^N (\tau_k - \tau_{k-1}) E(s_{\tau_{k-1}}^{(m)})^2 - \int_0^1 E(s_t^{(m)})^2 dt \right]^{1/2} \end{aligned}$$

by (H2), (H1) and (H3), we obtain therefore that (2.6) holds for $j=1$.

Turning our attention to $j=2$, we have:

$$\left| \sum_{k=1}^N (\bar{A}_k^{(m)} - A_{\tau_{k-1}}^{(m)}) (w_{\tau_k} - w_{\tau_{k-1}}) \right| \leq \|A^{(m)}\| \left(\sup_{\tau_k} |w_{\tau_k} - w_{\tau_{k-1}}| \right)$$

Therefore

$$|\tilde{J}_{N(M),2}^{(L)}| \leq \left(\sum_{m=M+1}^L |x^{(m)}| \|A^{(m)}\| \right) \sup_{\tau_k} |w_{\tau_k} - w_{\tau_{k-1}}| \leq 2 \sup_{0 \leq t \leq 1} |w_t| \sum_{m=M+1}^L |x^{(m)}| \|A^{(m)}\| \quad (2.8)$$

and to show the convergence in probability, note that

$$\begin{aligned} \sum_{m=M+1}^L E |x^{(m)}| \|A^{(m)}\| &\leq \sum_{m=M+1}^L E^{1/2}((A^{(m)})^2) E^{1/2}(x^{(m)})^2 \\ &\leq \sum_{m=M+1}^L E(A^{(m)})^2 + \sum_{m=M+1}^L E((x^{(m)})^2) \end{aligned} \quad (2.9)$$

which, by (H1), imply the convergence in probability to 0 of (2.8). Finally, turning to $j=3$, note that

$$E |\tilde{J}_{N(M),3}^{(L)}| \leq \sum_{m=M+1}^L E^{1/2}((x^{(m)})^2) E^{1/2} \left(\sum_{k=1}^N (\bar{M}_k^{(m)} - M_{\tau_{k-1}}^{(m)}) (w_{\tau_k} - w_{\tau_{k-1}}) \right)^2$$

$$\begin{aligned}
&\leq \sum_{m=M+1}^L E^{1/2}(x^{(m)})^2 \sum_{k=1}^N E^{1/2}[(\bar{M}_k^{(m)} - M_{\tau_k}^{(m)})^2 (w_{\tau_k} - w_{\tau_{k-1}})^2] \\
&\leq \sum_{m=M+1}^L E^{1/2}(x^{(m)})^2 \sum_{k=1}^N E^{1/4}[\bar{M}_k^{(m)} - M_{\tau_k}^{(m)}]^4 E^{1/4}[w_{\tau_k} - w_{\tau_{k-1}}]^4 \\
&\leq C \sum_{m=M+1}^L E^{1/2}(x^{(m)})^2 \sum_{k=1}^N E^{1/4} \left[\frac{\bar{M}_k^{(m)} - M_{\tau_k}^{(m)}}{(\tau_k - \tau_{k-1})^{1/2}} \right]^4 (\tau_k - \tau_{k-1})^2 \\
&\leq C' \sum_{m=M+1}^L E^{1/2}(x^{(m)})^2 b_m \sum_{k=1}^N (\tau_k - \tau_{k-1})^2
\end{aligned}$$

which, again by H1 and H2, ensure the convergence to zero (as $L \rightarrow \infty$ first and $N \rightarrow \infty$ next).

We next note that (A4) holds trivially, since due to the fact that $L < \infty$ it is enough to check that for each (m),

$$x^{(m)} \bar{I}_N^{(m)} \xrightarrow[N \rightarrow \infty]{\text{prob}} x^{(m)} I^{(m)} \text{ in probability}$$

which clearly holds since $\bar{I}_N^{(m)} \xrightarrow[N \rightarrow \infty]{\text{prob}} I^{(m)}$ since $s_t^{(m)}$ is adapted.

We finally check A3. Note that, by using the fact that $I^{(m)}$ is an adapted integral,

$$\begin{aligned}
E(|\bar{J}_{(M)}^{(L)}|) &\leq \sum_{m=M+1}^L E^{1/2}(x^{(m)})^2 E^{1/2}(I^{(m)})^2 \\
&\leq \sum_{m=M+1}^L (c_m^2 + b_m^2)^{1/2} E^{1/2}(x^{(m)})^2
\end{aligned} \tag{2.10}$$

which, by H1, H2 converges to zero when $M, t \rightarrow \infty$. The lemma is proved.

The following variant of lemma 2 is useful in the comparison with the Ogawa integral:

Lemma 3

Let (H1) - (H3) in Lemma 2 be replaced by:

$$c_m < K, \quad a_m < K, \quad b_m < K \tag{H1'}$$

$$\sum_{m=1}^{\infty} E^{1/2}((x^{(m)})^2) < \infty \tag{H2'}$$

$$\sum_{k=1}^N (\tau_k - \tau_{k-1}) E(s_{\tau_{k-1}}^{(m)})^2 < K \quad (H3')$$

then again $x_t \triangleq \lim_{m \rightarrow \infty} \left(\sum_{M=1}^M x_t^{(m)} s_t^{(m)} \right)$ exists (in $L_1(\Omega \times [0,1])$) and (A1-A5) still hold.

Proof: Repeat the proof of lemma 2, and note that in (2.7), (2.9), (2.10), (H1') and (H3') are still enough for convergence.

3. A Relation Between the Ogawa and the Stratonovich Integrals

In this section, we propose conditions under which the Stratonovich and Ogawa integrals both exist and coincide; these conditions unlike those in [1], [4] and references there do not make use of Malliavin-type calculus.

Let ϕ_n be any continuous, uniformly bounded orthonormal base of $L^2[0,1]$. We claim:

Lemma 4

Assume that $\sum_{k=1}^{\infty} \frac{1}{E^{2+\varepsilon}(y_k^{2+\varepsilon})} < \infty$, where $y_k \triangleq \int_0^1 \phi_k(t) y_t dt$. Then both the Ogawa and the

Stratonovich integrals of y_t exist and they are equal.

Proof. That the Stratonovich integrall exists follows from lemma 3 by taking $x^{(m)} = y_k$, $s_t^{(m)} = \phi_m$. That it is equal to the construction of the Ogawa integral for the basis ϕ_m is clear from definition 2, for

$$\begin{aligned} E\left(\sum_{k=M+1}^m y_k \int_0^1 \phi_k(s) dw_s\right)^2 &= \sum_{k,k'=M+1}^m E(y_k y_{k'}) \int_0^1 \phi_k(s) dw_s \int_0^1 \phi_{k'}(s) dw_s \leq \\ &\leq \sum_{k,k'=M+1}^m E^{\frac{1}{(1+\frac{\varepsilon}{2})}}((y_k y_{k'})^{1+\varepsilon/2}) E^{\frac{1}{(\frac{2}{\varepsilon}+1)}}\left(\int_0^1 \phi_k(s) dw_s\right)^{\frac{2}{\varepsilon}} \left(\int_0^1 \phi_{k'}(s) dw_s\right)^{\frac{2}{\varepsilon}+1} \end{aligned}$$

Note that $\int_0^1 \phi_k(s) dw_s$ are i.i.d. $N(0,1)$ random variable. Therefore, we get:

$$E\left(\sum_{M+1}^m y_k \int_0^1 \phi_k(s) dw_s\right)^2 \leq K \sum_{k,k'=M+1}^m \frac{1}{E^{2+\varepsilon}(y_k^{2+\varepsilon})} \frac{1}{E^{2+\varepsilon}(y_{k'}^{2+\varepsilon})}$$

$$= K \sum_{k=M+1}^m \frac{1}{E^{2+\varepsilon} (y_k^{2+\varepsilon})^2} \quad (3.1)$$

where K does not depend on m ; therefore, by our assumption, $\sum_1^m y_k \int_0^1 \phi_k(s) dw_s$ is a Cauchy sequence

in $L^2(\Omega)$, which implies that the sum in definition 2 converges and is therefore equal to the Stratonovich integral.

We need therefore to prove that the sum of definition 2 does not depend on the particular base chosen ϕ_n . Let ψ_n be another complete base, we therefore have to show that

$$A(K) \triangleq \sum_{k=1}^K (\psi_k, \xi) \int_0^1 \psi_k(t) dw_t + \sum_{k=1}^K (\phi_k, \xi) \int_0^1 \phi_k(t) dw_t \xrightarrow[K \rightarrow \infty]{L^2(\Omega)} 0 \quad (3.2)$$

where (ψ_k, ξ) denote the scalar product in $L^2[0,1]$.

Let $\psi_k = \sum_{n=1}^{\infty} \alpha_{nk} \phi_n$, where $\alpha_{nk} = (\psi_k, \phi_n)$, $\sum_{n=1}^{\infty} \alpha_{nk}^2 = 1$, $\sum_{k=1}^{\infty} \alpha_{nk}^2 = 1$.

Define

$$A(K,L,N) = \sum_{k=1}^K \sum_{n=1}^N \sum_{l=1}^L \alpha_{nk} \alpha_{lk} (\phi_n, \xi) \int_0^1 \phi_l dw_t - \sum_{n=1}^N (\phi_n, \xi) \int_0^1 \phi_n dw_t \quad (3.3)$$

Clearly, $A(K) \xrightarrow[K \rightarrow \infty]{L^2(\Omega)} 0$ if $\lim_{K \rightarrow \infty} \lim_{N \rightarrow \infty} \lim_{L \rightarrow \infty} E(A^2(K,L,N)) = 0$; Note that

$$\begin{aligned} A(K,L,N) &= \sum_{n=1}^K (\phi_n, \xi) \left[\sum_{k=1}^K \alpha_{nk} \sum_{l=1}^L \alpha_{lk} \int_0^1 \phi_l dw_t - \int_0^1 \phi_n dw_t \right] + \\ &+ \sum_{k=1}^K \sum_{n=K+1}^N (\phi_n, \xi) \alpha_{nk} \sum_{l=1}^L \alpha_{lk} \int_0^1 \phi_l dw_t \\ &\triangleq \sum_{n=1}^K (\phi_n, \xi) \sum_{k=1}^K B(n,k,L) + \sum_{n=K+1}^N (\phi_n, \xi) D(n,K,L) \triangleq A_1(K,L) + A_2(K,L,N) \quad (3.4) \end{aligned}$$

and we need to check the convergence of $A_i(K,L,N)$, $i = 1, 2$.

For $A_1(K,L)$, one has

$$\begin{aligned}
E(A_1^2(K,L)) &= E\left[\sum_{n=1}^K (\phi_n, \xi) \left(\sum_{k=1}^K B(n,k,L)\right)\right]^2 \\
&= E \sum_{n=1}^K \sum_{n'=1}^K (\phi_n, \xi) (\phi_{n'}, \xi) \left(\sum_{k=1}^K B(n,k,L)\right) \left(\sum_{k=1}^K B(n',k,L)\right) \\
&\leq \sum_{n=1}^K \sum_{n'=1}^K \frac{1}{E^{2+\varepsilon}((\phi_n, \xi)^{2+\varepsilon})} \frac{1}{E^{2+\varepsilon}((\phi_{n'}, \xi)^{2+\varepsilon})} \cdot \\
&\quad \cdot E^{\frac{\varepsilon}{4+2\varepsilon}} \left(\sum_{k=1}^K B(n,k,L)\right)^{\frac{4+2\varepsilon}{\varepsilon}} E^{\frac{\varepsilon}{4+2\varepsilon}} \left(\sum_{k=1}^K B(n',k,L)\right)^{\frac{4+2\varepsilon}{\varepsilon}} \\
&= \left(\sum_{n=1}^K \frac{1}{E^{2+\varepsilon}((\phi_n, \xi)^{2+\varepsilon})}\right) E^{\frac{\varepsilon}{4+2\varepsilon}} \left(\sum_{k=1}^K B(n,k,L)\right)^{\frac{4+2\varepsilon}{\varepsilon}}^2
\end{aligned} \tag{3.5}$$

Note that $B(n,K,L)$ is a Gaussian r.v., mean zero, and

$$\lim_{L \rightarrow \infty} E(B(n,K,L))^2 = 1 - \sum_{k=1}^K \alpha_{nk}^2 \tag{3.6}$$

Therefore,

$$\lim_{L \rightarrow \infty} E(A_1^2(K,L,N)) \leq c(\varepsilon) \sum_{n=1}^K \frac{1}{E^{2+\varepsilon}((\phi_n, \xi)^{2+\varepsilon})} \left(1 - \sum_{k=1}^K \alpha_{nk}^2\right)^{1/2} \xrightarrow{K \rightarrow \infty} 0$$

Turning to $A_2(K,L,N)$, note that $D(n,K,L)$ is again a Gaussian r.v., mean zero and

$$\lim_{L \rightarrow \infty} E(D^2(n, K, L)) = \sum_{k=1}^K \alpha_{nk}^2 \tag{3.7}$$

and therefore, repeating the computations above, one has:

$$\lim_{L \rightarrow \infty} E(A_2^2(K,L,N)) \leq c(\varepsilon) \sum_{n=K+1}^N \frac{1}{E^{2+\varepsilon}((\phi_n, \xi)^{2+\varepsilon})} \left(\sum_{k=1}^K \alpha_{nk}^2\right)^{1/2} \leq c(\varepsilon) \sum_{n=K+1}^N E^{1(2+\varepsilon)}(\phi_n, \xi)^{2+\varepsilon} \tag{3.8}$$

and, from (3.8) and our conditions in the statement of the lemma, we again get the required convergence, which completes the proof of the lemma.

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